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ELASTIC–PLASTIC TORSION AS A PROBLEM IN NON-LINEAR PROGRAMMING*

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Abstract—For a simply connected region, it is shown that the problem of determining the elastic–plastic stress function is equivalent to that of minimizing the complementary energy subject to the inequality constraints required by the yield condition. A method is proposed for approximating this minimum in which the cross section is approximated by a finite number of triangles and a linear stress function is assumed for each triangle. This approximate problem is then solved by means of a recently available computer program for solving nonlinear constrained minimization problems.

STATEMENT OF THE PROBLEM

WE consider a simply connected cylindrical or prismatic bar which has a cross section A bounded by a curve C, and take the x_3 axis to be parallel to the generators of the bar. Then, as is well known, the complete state of the bar under Saint Venant torsion is defined by the two non-vanishing stress components σ_{31}, σ_{32} and the warping function w, all quantities being functions of x_1 and x_2.

To formulate the problem in dimensionless terms, let B denote a typical length in the cross section, L the length of the bar, k the material yield stress in pure shear, and G the elastic shear modulus. Dimensionless coordinates are defined by

$$x = x_1/B \qquad y = x_2/B \tag{1}$$

and a dimensionless angle of twist by

$$\theta = 2GB\alpha/k \tag{2}$$

α being the twist per unit length.

For any material, equilibrium requirements in the interior and on the boundary are automatically satisfied if the stresses are derived from a dimensionless stress function ψ by

$$\sigma_{31} = k\partial\psi/\partial y, \qquad \sigma_{32} = -k\partial\psi/\partial x \tag{3}$$

where

$$\psi \text{ is continuous and piecewise continuously differentiable in } A \tag{4}$$

and

$$\psi = 0 \qquad \text{on } C. \tag{5}$$

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If the material is elastic–perfectly-plastic and the angle of twist is non-decreasing, the stress function must satisfy the further conditions [1]

$$|\nabla\psi| \leq 1 \quad \text{in } A \tag{6}$$

$$\text{if } |\nabla\psi| < 1, \qquad \nabla^2\psi + \theta = 0 \quad \text{in } A \tag{7}$$

$$\psi \text{ continuously differentiable in } A. \tag{8}$$

For a perfectly elastic material, equation (6) and the condition in equation (7) are irrelevant, so that equation (7) must hold throughout the bar. The complementary energy π_c of the bar is then [2] $(LB^2k^2/2G)\Pi_c$ where

$$\Pi_c = \int_A (|\nabla\psi|^2 - 2\theta\psi)\,\mathrm{d}A \tag{9}$$

Further, if we consider the class of stress functions ψ^0 which satisfy equations (3)–(5), the elastic theorem of minimum complementary states that among all functions of ψ^0 the actual solution satisfying equations (7) and (8) minimizes Π_c.

We assume throughout that θ is a monotonically non-decreasing function of time. It has been shown [1] that at any point P in A, if ψ satisfies the equality in equation (6) for some angle θ_0, then ψ is constant in time for all $\theta > \theta_0$. Under these conditions, it follows as a special case of a theorem proved in [2] for an elastic–perfectly-plastic material that (a) Π_c is still given by (9) and (b), if the class of statically admissible stress functions is further restricted by (6), then the principle of minimum complementary energy is still valid.*

This minimum principle is the basis of the numerical method presented in the following section. A function ψ which depends upon several parameters ψ_k is defined so that equations (4) and (5) are automatically satisfied for any choice of ψ_k, and the ψ_k are then chosen so as to minimize Π_c subject to equation (6).

For an elastic material, it is well known that any approximate solution obtained in this way can be used to obtain a lower bound on the torque necessary to produce a given angle of twist. However, as is pointed out in Appendix A, this interpretation is not necessarily valid for the elastic–perfectly-plastic material considered here.

METHOD OF SOLUTION

Various methods can be used to choose the nature of the dependence of ψ upon its parameters. A usual technique is to choose them as the coefficients of a complete set of functions such as a Fourier series, and then to consider a finite subset. This approach has a great theoretical advantage in that it may be possible to prove that the approximate solution can be made arbitrarily close to the exact solution by choosing a sufficient number of parameters. However, for an irregularly shaped boundary, automatic satisfaction of the boundary condition (5) may be far from trivial. Also, as will be discussed shortly, special problems are posed by a constrained minimization problem [ψ must satisfy equation (6)].

A second method, recently used by Koopman and Lance [4] in a plasticity problem, is to begin by replacing the continuous region by a finite array of mesh points, and the

* This fact was first conjectured by Haar and von Karman [3].

differential equations by finite difference equations. The values of ψ at the interior mesh points are taken as the parameters ψ_k. Applied to the present situation, equation (9) for evaluating Π_c would be replaced by a finite integration formula to yield a value Π_c'. Also, the inequality constraint (6) would be replaced by finite difference approximations. Alternatively, we could take advantage of the Nadai analogy [5, 2] and replace (6) by

$$\psi_k \leq \bar{\psi}_k \tag{10}$$

where $\bar{\psi}_k$ are the easily determined values of the fully plastic stress function at the mesh points.

However, both of the above methods are open to the serious objection that in practice they may not yield valid answers. The minimum principle is known to be valid only if the approximate solution satisfies (6) at every point of A, whereas in either of the schemes mentioned, (6) can be enforced only over a finite subset of A. This objection might not be serious if it were not for the weak continuity requirements on ψ imposed by equation (4). Thus, the derivatives of ψ which are required for (6) may not even be continuous. Although a Fourier series can give, in a total sense, a good approximation to a discontinuous function, the approximation may be very poor sufficiently close to the discontinuity. Figure 1 illustrates qualitatively what may happen. If (6) is checked at points A and C it will appear

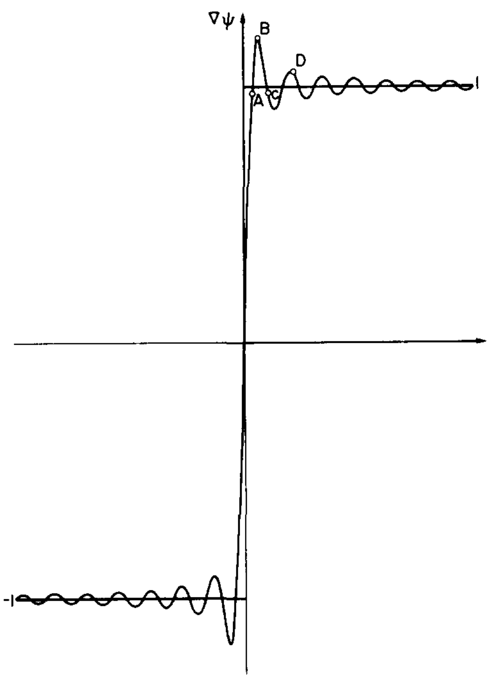

FIG. 1. Fourier series representation of a discontinuous function.

to be satisfied but will, in fact be drastically violated at B so that the function is far from statically admissible. Even if it is checked at D, it would appear that division by a factor only slightly greater than 1 should result in a statically admissible field, whereas in fact (6) would still be badly violated at B. On the other hand, if it is checked at B, the necessary division factor will be much larger than 1, (6) will be strongly satisfied everywhere else, and the resulting lower bound will almost certainly be a poor one.

The same objection applies to the second method, although not so seriously since (6) can be replaced by (10) which involves only a continuous function. However, the continuity requirements (4) pose a further difficulty in that it is not easy even to define continuity of a function specified only over a finite set of points.

The method used in the present paper was first used by Herrmann [6] in a purely elastic problem. It may be considered as a special case of finite difference approach, chosen in such a way as to eliminate the difficulties of the general case. Briefly, the region A is replaced by a set of triangles, together with such curvilinear regions as may be left over at the boundary. At each vertex in the interior of A ψ is assigned a parameter value ψ_k, at each boundary vertex ψ is assigned the value 0, ψ is defined to be linear in the interior of each triangle, and ψ is identically zero in each left over curvilinear region.

Evidently, for any set of values of the parameters ψ_k, the function ψ is uniquely defined. Further, it obviously satisfies the continuity requirements (4) rigorously in A, and the boundary conditions (5) everywhere on C. Also, since ψ is linear within each triangle its gradient is a constant and satisfaction of (6) at every point of the triangle is equivalent to a single quadratic inequality in terms of the parameters ψ_k of its three vertices. Finally, Π_c can be evaluated in closed form so that no approximate integration scheme is needed. Therefore, the minimum principle is rigorously valid and any approximate solution so obtained will provide an upper bound on Π_c for the actual solution.

TECHNIQUE OF SOLUTION

We divide the cross section of the cylinder into m triangles with n interior vertices and n_b boundary vertices. Consider an arbitrary triangle and let its vertices be temporarily denoted by the subscripts, i, j, k. Then, in that triangle, $\psi(x, y)$ is the linear function

$$\psi(x, y) = (x\partial\psi/\partial x + y\partial\psi/\partial y + D) \tag{11}$$

where

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= \frac{1}{\Delta}\begin{vmatrix} \psi_i & y_i & 1 \\ \psi_j & y_j & 1 \\ \psi_k & y_k & 1 \end{vmatrix} \qquad \frac{\partial\psi}{\partial y} = \frac{1}{\Delta}\begin{vmatrix} x_i & \psi_i & 1 \\ x_j & \psi_j & 1 \\ x_k & \psi_k & 1 \end{vmatrix} \\
D &= \frac{1}{\Delta}\begin{vmatrix} x_i & y_i & \psi_1 \\ x_j & y_j & \psi_2 \\ x_k & y_k & \psi_3 \end{vmatrix} \qquad \Delta = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}
\end{aligned} \tag{12}$$

Since ψ is linear, its derivatives are constant, and hence equation (9) is trivially integrated, in each triangle. Summing over the m triangles, we obtain

$$\Pi_c = \sum_{l=1}^{m} |\Delta| \left\{ \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] - \theta\frac{\psi_i+\psi_j+\psi_k}{3} \right\} \tag{13}$$

The problem is to minimize (13) subject to the m inequalities

$$g_l \equiv 1-[(\partial\psi/\partial x)^2+(\partial\psi/\partial y)^2] \geq 0 \qquad l = 1,\ldots,m \tag{14}$$

where all quantities are defined in terms of the coordinates and parameter values of the triangle vertices by equations (12).

This problem has been programmed for an IBM 7040 using FORTRAN IV and a SHARE Library program [7] entitled "Sequential Unconstrained Minimization Technique" (SUMT). The method was first suggested by Carroll [8]; its validity was proved by Fiacco and McCormick [9] who also wrote the SHARE program. The essential idea is as follows:

We define an auxiliary "primal" function by

$$P(\boldsymbol{\psi}, r) = \Pi_c + r \sum_{l=1}^{m} 1/g_l \tag{15}$$

where $\boldsymbol{\psi} \equiv (\psi_1,\ldots,\psi_n)$ is a vector with n components. A "feasible point" is a vector $\boldsymbol{\psi}$ which satisfies all of the *strict* inequalities (14), and the "feasible domain" is the totality of feasible points. For the particular problem under consideration it is evident that the feasible domain is strictly convex and contains the point $\boldsymbol{\psi} = 0$, and that Π_c is a strictly convex function of $\boldsymbol{\psi}$. Further, P is bounded from below and finite at any point Q in the feasible domain, but tends to plus infinity as Q tends to the boundary of the feasible domain.

For any given value of r, it follows from the above that P will have a unique minimum

$$\bar{P}(r) = P[\overline{\boldsymbol{\psi}}(r), r] \tag{16}$$

at a point $\overline{\boldsymbol{\psi}}$ in the feasible domain. Further, it can be shown [9] that as r tends to zero, $\bar{P}(r)$ tends to the desired minimum value of Π_c and $\overline{\boldsymbol{\psi}}(r)$ tends to the desired vector which minimizes (13) subject to (14).

A program EPT1 was written to solve the Elastic–Plastic Torsion problem for any simply connected cross section. The user must write a short subroutine to describe the cross section and must provide a few data cards giving an initial subdivision into triangles and various accuracy standards. The program first solves the elastic problem for the given triangle-approximation, then subdivides each triangle one or more times for a more accurate result. Next the fully plastic problem is solved, and then elastic–plastic problems for given increments of θ until the torque is sufficiently close to the fully plastic value. All the above solutions are obtained by calling upon the SUMT program.

Details of the computer program, including a listing and analysis of output are described in Appendix B. This appendix has been bound separately as DOMIIT-Report 1-33A, Illinois Institute of Technology and may be obtained from the author on request. At the same time, the interested reader should request Ref. [7] either from the SHARE Library or from Dr. G. P. McCormick at the Research Analysis Corporation, McLean, Virginia.

EXAMPLES

To test the program, four examples were considered as shown in Fig. 2. The figure also illustrates the extent to which symmetry was used and the initial division into 3 triangles. After a first estimate of the elastic solution was obtained, each triangle was divided into four similar triangles by joining the midpoints of each side. This process was repeated until eventually 48 triangles were obtained for the circle, oval, and cruciform, and 192 for the triangle.

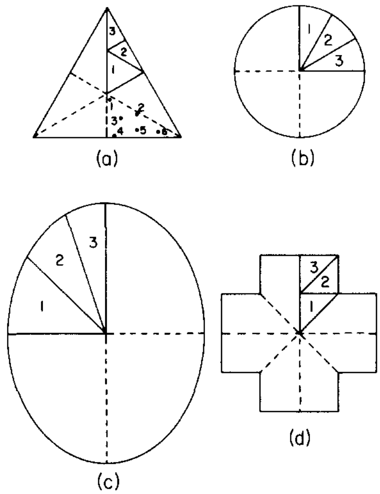

FIG. 2. Cross-sections studied, symmetry assumed, initial division into 3 triangles: (a) equilateral triangle; (b) circle; (c) Sokolovsky oval; (d) cruciform.

The equilateral triangle was solved only for the fully elastic solution. Table 1 indicates how the values of the stress function and the magnitude of the gradient vary with the number of triangles at the indicated points in Fig. 2(a). Also shown is the computed torque and the

TABLE 1. VALUES OF STRESS FUNCTION AND GRADIENT AT TYPICAL POINTS SHOWN IN FIG. 2 ($\theta = 1{\cdot}0$)

Point	1	2	3	4	5	6
Triangles			Stress function $\times 10^{-4}$			
3	8333	5208	4166	0	1389	1389
12	8329	5611	5722	0	2135	1350
48	8332	5698	6118	0	2369	1391
192	8334	5697	6217	0	2422	1396
Exact	8333	5697	6250	0	2440	1399
			Gradient magnitude $\times 10^{-4}$			
3	1667	1667	1667	1667	1667	1667
12	751	1456	1462	2907	2934	1234
48	365	1321	1535	3381	2336	1441
192	181	1210	1489	3578	2648	1337
Exact	181	1220	1502	3640	2546	1372
		Torque $\times 10^{-5}$			Minutes	
3		1202			0·04	
12		1514			0·84	
48		1596			3·96	
192		1617			46·08	
Exact		1624				

machine running time in minutes. The exact solution is well known (cf.[10]) and is shown for comparison. The values of the stress function and the torque are quite reasonably approximated with 48 triangles; although the gradient values are less accurate, the location of maximum gradient magnitude is correctly picked out. Because of the tremendous increase in machine time necessary to solve for 192 triangles, all other examples were stopped with 48.

Figure 3 shows the stress function for the circle for various angles of twist. The exact solution [10]

$$\begin{aligned} \psi &= (\theta/4)(1-r^2) && \theta \leq 2 \\ \psi &= 1-1/\theta-\theta r^2/4 && \theta \geq 2, \quad 0 \leq r \leq 2/\theta \\ \psi &= 1-r && \theta \geq 2, \quad 2/\theta \leq r \leq 1 \end{aligned} \tag{17}$$

is also shown. The double curves indicate that the approximate solution is not strictly a function of r only.

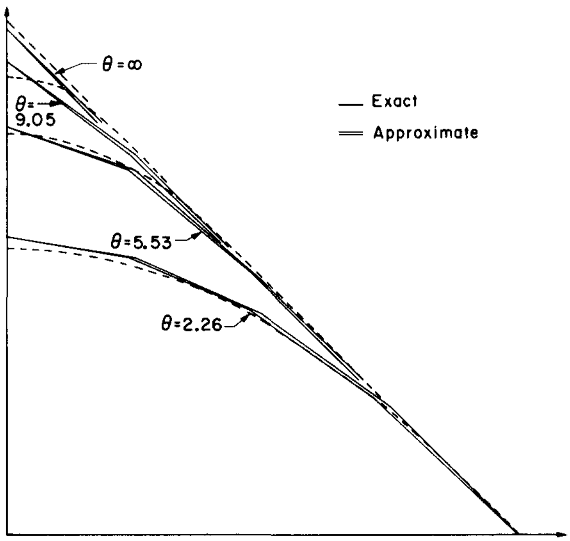

FIG. 3. Elastic–plastic stress function for circle.

The third example, the Sokolovsky oval [10, 11], is one of the few nontrivial sections for which the exact elastic–plastic solution is known. Figure 4 shows the exact and approximate elastic–plastic boundaries for various angles of twist. The fact that a triangle jumps as a unit from the elastic to the plastic state is responsible for the rather erratic approximate boundaries, but despite this fact there is reasonable qualitative agreement.

The final example of a cruciform cylinder indicates that reentrant corners pose no difficulties. Figure 5 shows the motion of the (approximate) elastic–plastic boundary. Although an exact solution is not known, the results are at least qualitatively correct in that plastic regions begin at the reentrant corners (but only for a finite angle of twist) and grow out; at a greater angle of twist additional plastic regions start at the ends of the legs; and for large angles of twist the elastic region appears to be shrinking down on the lines of discontinuity of the fully plastic solution.

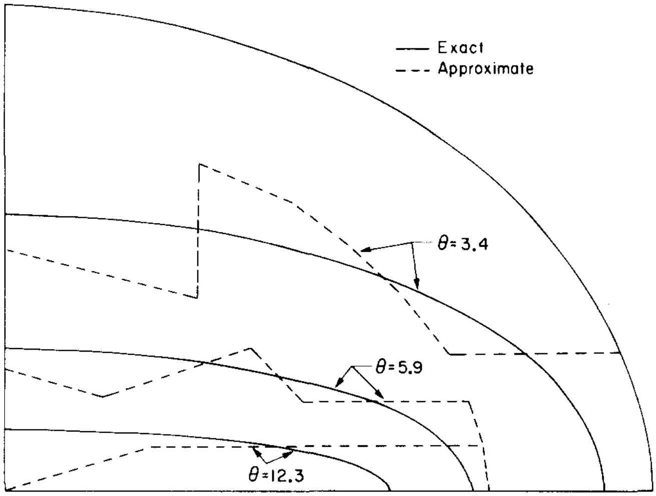

FIG. 4. Elastic–plastic boundaries for Sokolovsky oval.

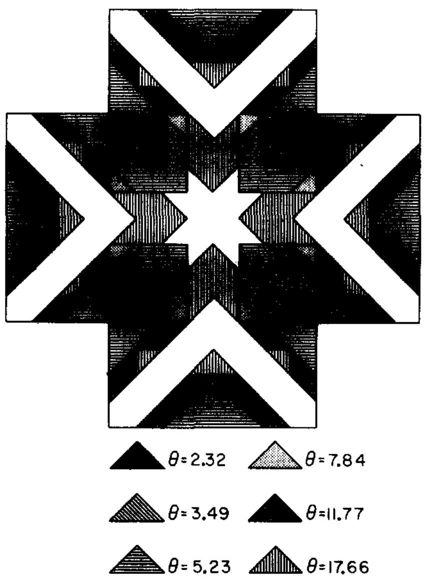

FIG. 5. Growth of plastic region for cruciform.

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APPENDIX A

DIRECT PROOF OF MINIMUM PRINCIPLE

Although we are giving a direct proof for the particular case of torsion, it is convenient to begin with the general definition of the complementary energy. Let V be a three dimensional region whose surface S is composed of a part S_D on which in each of three independent directions either the displacement is prescribed or the traction is zero, and a part S_T on which the roles of displacements and tractions are reversed. Then the complementary energy is defined by

$$\pi_c = \int_V U_c \, \mathrm{d}V - \int_{S_D} \mathbf{T} \cdot \mathbf{u} \, \mathrm{d}S \tag{A1}$$

where $\mathbf{T}$ is the traction vector, $\mathbf{u}$ the displacement vector, and

$$U_c = \int \boldsymbol{\varepsilon} : \mathrm{d}\boldsymbol{\sigma}. \tag{A2}$$

For the torsion problem, the only non-vanishing stresses are $\sigma_{\alpha 3} = \sigma_{3\alpha}$ where Greek subscripts have the range 1, 2. Denoting the corresponding engineering strains by γ_α, we may write the constitutive equations for an elastic–perfectly-plastic material in the form

$$G\dot{\gamma}_\alpha = \dot{\sigma}_{\alpha 3} + \dot{\lambda}\sigma_{\alpha 3} \tag{A3}$$

where $\dot{\lambda}$ is a non-negative scalar which is zero except during plastic flow. It is known [1] that once a material point of a simply connected cylinder becomes plastic, the stresses remain constant under monotonic torque. Therefore (A3) can be integrated with respect to time to yield

$$G\gamma_\alpha = (1+\lambda)\sigma_{\alpha 3} \tag{A4}$$

where λ is zero in the elastic range and non-negative if the point is plastic.

Substituting (A4) into (A2) we can write the result as

$$U_c = (1/G)\int (1+\lambda)\sigma_{\alpha 3} \, \mathrm{d}\sigma_{\alpha 3} = \sigma_{\alpha 3}\sigma_{\alpha 3}/2G \tag{A5}$$

The last step in (A5) follows from the facts that λ is zero in the elastic range and $\mathrm{d}\sigma_{\alpha 3}$ vanishes when the point is plastic. Therefore, for the torsion problem, the complementary energy is

$$\pi_c = (1/2G)\int_V \sigma_{\alpha 3}\sigma_{\alpha 3} \, \mathrm{d}V - \int_{S_D} \mathbf{T} \cdot \mathbf{u} \, \mathrm{d}S. \tag{A6}$$

To show that the actual solution minimizes π_c, we define π_c^0 analogously to (A6) for any statically admissible stress state and form the difference

$$\Delta\pi_c \equiv \pi_c^0 - \pi_c = (1/2G)\int_V (\sigma_{\alpha3}^0\sigma_{\alpha3}^0 - \sigma_{\alpha3}\sigma_{\alpha3})\,\mathrm{d}V - \int_{S_D} (\mathbf{T}-\mathbf{T}^0)\,.\,\mathbf{u}\,\mathrm{d}S. \tag{A7}$$

Since the second integrand in (A7) vanishes on S_T, we may equally well take the integral over the entire surface. We can then apply the principle of virtual work to obtain

$$\int_{S_D} (\mathbf{T}-\mathbf{T}^0)\,.\,\mathbf{u}\,\mathrm{d}S = \int_V (\sigma_{\alpha3}^0 - \sigma_{\alpha3})\gamma_\alpha\,\mathrm{d}V \tag{A8}$$

Next, we substitute (A8) into (A7) and use (A4) to obtain

$$\Delta\pi_c = (1/G)\int_V [\tfrac{1}{2}(\sigma_{\alpha3}^0 - \sigma_{\alpha3})(\sigma_{\alpha3}^0 - \sigma_{\alpha3}) + \lambda\sigma_{\alpha3}(\sigma_{\alpha3} - \sigma_{\alpha3}^0)]\,\mathrm{d}V \tag{A9}$$

At an elastic point λ vanishes, and at a plastic point we must have λ non-negative and

$$\sigma_{\alpha3}\sigma_{\alpha3} = k^2 \qquad \sigma_{\alpha3}\sigma_{\alpha3}^0 \leq k^2 \tag{A10}$$

so that the second term in (A9) is non-negative at every point. Since the first term is the sum of squares, we have shown that $\pi_c^0 \geq \pi_c$, QED.

It remains to show that the expressions (A6) and (9) for the complementary energy are equivalent. For the torsion problem, S_D is the end $x_3 = L$ of the cylinder where T_3 vanishes and

$$u_1 = -\alpha L x_2 \qquad u_2 = \alpha L x_1. \tag{A11}$$

Therefore, introducing the stress function from equation (3) and converting to the dimensionless variables defined in the first section, we may write

$$\int_{S_D} \mathbf{T}\,.\,\mathbf{u}\,\mathrm{d}S = \int_{S_D} \sigma_{\alpha3}u_\alpha\,\mathrm{d}S = -(k^2B^2L\theta)/2G\int_A \mathbf{x}\,.\,\nabla\psi\,\mathrm{d}A. \tag{A12}$$

Further, since $\sigma_{\alpha3}$ is independent of x_3 we may write

$$\int_V \sigma_{\alpha3}\sigma_{\alpha3}\,\mathrm{d}V = k^2B^2L\int_A |\nabla\psi|^2\,\mathrm{d}A. \tag{A13}$$

Finally, integrating the last integral in (A12) by parts by means of the divergence theorem, making use of the boundary condition (5), and substituting (A12) and (A13) into (A6), we obtain

$$\pi_c = (k^2B^2L/2G)\int_A [|\nabla\psi|^2 - 2\theta\psi]\,\mathrm{d}A \tag{A14}$$

in agreement with equation (9).

The torque applied to the end of the bar is readily computed to be

$$T = 2kB^3\int_A \psi\,\mathrm{d}A. \tag{A15}$$

Also, we can integrate (A14) by parts and use the boundary condition (5) to obtain

$$\pi_c = -(k^2B^2L/2G)\int_A (\nabla^2\psi+2\theta)\psi\,\mathrm{d}A. \tag{A16}$$

For a fully elastic bar, it then follows from (7) that the torque is proportional to $-\pi_c$, so that an upper bound on π_c can immediately be interpreted as a lower bound on T. However, if any part of the bar is plastic, equation (7) is not applicable, and hence the bound cannot be so interpreted.

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Résumé—Pour une région à assemblage simple, il est démontré que le problème de déterminer la fonction de tension élastique plastique équivaut à celui de minimiser l'énergie complémentaire soumise aux contraintes d'inégalité nécessitées par l'état de la limite d'élasticité. Une méthode est proposée pour obtenir une valeur approximative de ce minimum dans laquelle la section est représentée par une valeur approximative obtenue à partir d'un nombre fini de triangles et l'on suppose une fonction de tension linéaire pour chaque triangle. Ce problème d'approximation est ensuite résolu au moyen d'un programme par machine à calculer récemment rendu disponible pour la solution de problèmes de minimisation de contraintes non linéaires.

Zusammenfassung—Es wird gezeigt, dass für einen verbundenen Bereich das Problem der Bestimmung der elasto-plastischen Spannungsfunktion dem der Verkleinerung der Komplementärenergie gleichwertig ist sofern diese den Ungleichheitsbedingungen der Fliessbedingungen folgen. Eine Annäherungsmethode für das Minimum wird vorgeschlagen, wobei der Durchschnitt durch eine endliche Anzahl von Dreiecken bestimmt wird und die lineare Spannungsfunktion für jedes Dreieck angenommen wird. Dies Problem wird dann mittels eines neuerlich vorhandenen Rechnerprogrammes, für die Lösung nichtlinearer Minimisationsprobleme, gelöst.

Абстракт—Оказывается, что для простого связанного района, задача об определении упруго-пластической функции напряжения эквивалентна минимализации полной энергии лредмета к нераренству сил связи, вытекающей из условия течения. Строится метод аппроксимации такого минимума, в котором приближается поперечное сечение конечным числом трехугольников. Подсчитывается линейная функция напряжения для каждого трехугольника. Далее решается эту приближенную задачу с помощью доступной в последнее время программы для счетных машин. Эта программа составлена для расчета нелинейных, несвобдных минимализационных задач.